Impact Pressure Probe in Free Molecular Flow

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The impact pressure probe in free molecular flow is studied theoretically on the assumption that the gas entering the tube has a Maxwellian distribution and that the molecules are reflected diffusely at the tube wall. The case is considered in which the tube length is very large compared with its diameter. An asymptotic theory for a long tube is employed in the analysis to obtain the pressure at the gage volume attached to one end of the tube. Numerical calculations are made on the properties of the probe for speed ratios up to 5.0 and angles of attack ranging from 0 to 180 deg. A comparison with previous work based on the Clausing approximation makes it clear that the error due to this approximation increases with the speed ratio, reaching about 10% at a speed ratio of 5.0. An explicit formula for the gage pressure is also obtained in the case of small speed ratios.

Nomenclature

$\hat{a}$	= radius of probe
A	= length to diameter ratio $[=L/(2\hat{a})]$
C_i, D_i	= variational parameters
F	= number flux
k	= Boltzmann's constant
L	= length of probe
m	= mass of a molecule
n	= number density
p	= pressure
Q, R	= mean impact density
r, ψ, x	= cylindrical coordiantes
S	= speed ratio $[=V_0/(2kT_0/m)^{1/2}]$
T	= temperature
V	= mean velocity
α	= angle of attack
ξ	= stretched coordinate $(=Ax)$

Subscripts

0	= values in uniform flow
c	= gage values

I. Introduction

THE impact pressure probe is a device to deduce the speed in free molecular flow by measuring the pressure in the gage volume attached to the end of a tube. Significant theoretical and experimental work[1-7] has been performed in investigating the properties of the probe, i.e., the dependence of the gage pressure upon the speed ratio, the angle of attack of the probe, and the ratio of length to radius of the tube. Perhaps a first step in investigating this problem is in obtaining the so-called escape probability[8] that a molecule leaving the tube wall will eventually reach the outlet of the tube. This probabiltity is satisfied by a Fredholm integral equation of the second kind. The flux of molecules traveling through the tube to the gage volume can be computed, once the escape probability and the number of molecules along the wall of the tube arising from the direct impact from the freestream are obtained. Then, the gage pressure, which depends on the flux, can be calculated. All theoretical work was based on the fundamental assumption that a solution of the integral equation concerned could be approximately expressed as a linear function of the distance along the tube axis. This approximation was first proposed by Clausing,[8] concerning the flow through a tube from a reservoir to a vacuum. It was shown that the number flux obtained with the approximate solution is fairly good.[8-10] It has been expected that the Clausing approximation would give good results even to a case like an impact pressure probe where the gas flows with a mean velocity past a tube.

In the present study, the same problem is investigated in a different way without assuming the Clausing approximation. The problem is formulated in terms of the impact density,[11] which is the number of molecules leaving the wall per unit area and per unit time. The impact density satisfies an integral equation similar to that of the escape probability. A case is considered of a long tube in which the Clausing approximation might produce a large error. An asymptotic theory for a large length-radius ratio of the tube is employed to obtain the gage pressure. A variational method is used in evaluating the quantities involved. Numerical calculations are made on the gage pressure for a wide range of speed ratios and angles of attack of the probe. In the case of small speed ratios, an explicit formula for the pressure in the gage volume is also derived.

II. Formulation of the Problem

Consider a circular cylindrical impact pressure probe which is connected to a large gage volume. The Maxwellian gas having a mean velocity V_0, a temperature T_0, and a density n_0 flows at an angle α with the tube axis (Fig. 1). It is assumed that the pressure is so low that there is no collision between molecules. It is also assumed that the molecules are reflected diffusely from the wall surface. Since the flow is collisionless, the molecule entering the tube from the freestream proceeds without interference with the molecule coming from the pressure gage; therefore these two can be treated separately. First, we consider the molecules from the freestream. Some hit a point on the tube wall directly, while others reach there after one or more reflections. The number flux normal to the wall should be zero; for there is no absorption or emission of the molecules. The conservation of molecules at the wall leads to an integral equation for the impact density $n_0[kT_0/(2\pi m)]^{1/2}\hat{Q}(x,\psi)$. Now, the mean impact density $Q(x)$ is introduced and is defined by

$$Q(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{Q}(x,\psi)\,\mathrm{d}\psi$$

Then, $Q(x)$ is found to be satisfied by the following equation:

$$A^{-1}Q(x)=N(x;S,\alpha)+\int_0^1 Q(x_1)f(A\,|x-x_1|)\mathrm{d}x_1 \tag{1}$$

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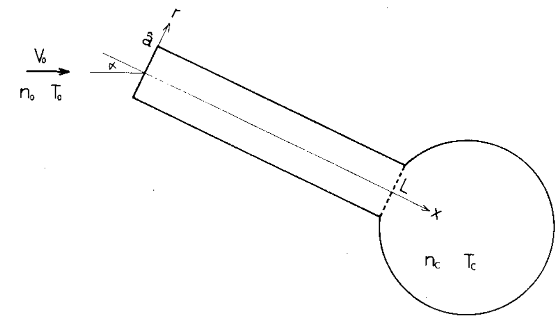

Fig. 1 Configuration and coordinate system.

where

$$f(x) = 1 - x/\sqrt{1+x^2} - \tfrac{1}{2}x/(1+x^2)^{3/2} \tag{1a}$$

$$N(x;S,\alpha) = \frac{e^{-S^2}}{16\pi^2}\int_{-\pi}^{\pi} d\psi \int_{-\infty}^{0} dx_1 \int_{-\pi}^{\pi} \{1+G^2 + \sqrt{\pi}e^{G^2}(G^3+3/2G)[1\pm \mathrm{erf}(|G|)]\}K d\psi_1 \tag{1b}$$

the plus sign in the bracket corresponding to $G>0$ and the minus sign to $G<0$ and

$$G = S\frac{\{A(x-x_1)\cos\alpha + \tfrac{1}{2}(\cos\psi - \cos\psi_1)\sin\alpha\}}{\{A^2(x-x_1)^2 + \tfrac{1}{2}[1-\cos(\psi-\psi_1)]\}^{1/2}} \tag{1c}$$

$$K = \frac{[1-\cos(\psi-\psi_1)]^2}{\{A^2(x-x_1)^2 + \tfrac{1}{2}[1-\cos(\psi-\psi_1)]\}^2} \tag{1d}$$

The number flux through the tube $\pi a^2 n_0 [kT_0/(2\pi m)]^{1/2} F$, which may be evaluated at the entrance of the tube ($x=0$) or at the midsection ($x=\tfrac{1}{2}$), can be given by

$$F = 4A\{A\int_0^{\infty} N(x;S,\alpha)dx + \int_0^1 Q(x) f_1(Ax)dx\} \tag{2}$$

$$F = 4A\{A\int_{1/2}^{\infty} N(x;S,\alpha)dx - \int_0^{1/2} Q(x) f_1\left[A\left(\frac{1}{2}-x\right)\right]dx + \int_{1/2}^{1} Q(x) f_1\left[A\left(x-\frac{1}{2}\right)\right]dx\} \tag{2a}$$

where

$$f_1(x) = x - \sqrt{1+x^2} + 1/(2\sqrt{1+x^2}) \tag{2b}$$

The counterflow, i.e., the flow from the pressure gage, can be formulated in a similar way. It is assumed that the molecules entering the tube from the gage have a Maxwellian distribution with zero mean velocity, a constant temperature T_c and a density n_c. Obviously, if we put $S=0$ in Eqs. (1) and (2a) we obtain the equations for the mean impact density $n_c[kT_c/(2\pi m)]^{1/2} R(x)$ and the number flux $\pi a^2 n_c [kT_c/(2\pi m)]^{1/2} F_c$, respectively. Carrying out the integration involved in $N(x;0,\alpha)$, we get from Eq. (1)

$$R(x) = -f_1(Ax) + A\int_0^1 R(x_1) f(A|x-x_1|)dx_1 \tag{3}$$

It may be said that this equation is nothing but the equation of escape probability.

Equation (2), which gives the number flux F, includes the integral of $Q(x)$ which, in principle, could be obtained by solving Eq. (1). It will be shown that this can be transformed to the integral involving $R(x)$ in the following way: $f_1(Ax)$ in Eq. (3) is substituted in the second integral of Eq. (2) and there, $Q(x)$ is eliminated through Eq. (1). The following equation is then obtained

$$\int_0^1 Q(x) f_1(Ax)dx = -A\int_0^1 R(x) N(x;S,\alpha)dx \tag{4}$$

$$\to \int_0^1 R(x) f_1(Ax)dx \quad \text{for } S=0 \tag{4a}$$

This relation shows that the number flux F may be calculated with a solution $R(x)$ of Eq. (3) together with $N(x;S,\alpha)$, which is the mean number of molecules on the wall arising from direct impact from the freestream. Clausing[8] proposed a simple approximate solution of $R(x)$, which has the following form:

$$R(x) \simeq H(A) + [1-2H(A)]x$$

where H is a function of A and was taken in such a way that the approximate $R(x)$ gives correct values of the integral in Eq. (4a) for both limits, $A=0$ and $A\to\infty$ when $S=0$. Previous theoretical work on the impact probe[2-4,6,7] in which $S\neq 0$ is mostly based on the Clausing solution. In the present study, however, neither Eqs. (3) and (4) nor Clausing's approximation are used. Equations (1) and (2a) are used instead, developing more rigorous analysis.

A formula is added here for the pressure ratio. That the total number flux of molecules through the tube should be zero in a steady state leads to the formula for the pressure ratio expressed by the flux ratio:

$$p_c/p_0 = (T_c/T_0)^{1/2}(F/F_c) \tag{5}$$

III. Analysis

Most Regions of Long Tubes, Excluding End Regions

The analysis is restricted to long tubes, for which $A \gg 1$ whereas S is finite. The asymptotic theory developed in Refs. 11 and 12 may be applied to solve Eq. (1). First, the behavior of the solution in most of the tube where both x and $1-x \gg 1/A$ is considered. The nonhomogeneous term in Eq. (1) can be expanded there. The kernel $f(A|x-x_1|)$ of Eq. (1) decreases to zero very rapidly outside the range of $|x-x_1| \sim O(1/A)$. On the other hand, the unknown function $Q(x_1)$ may be a slowly varying function, and hence it may be expanded in a Taylor series around $x_1 = x$. After some manipulation, an asymptotic equation is obtained, which is easily solved by successive approximation to give

$$Q(x) = \beta x + \gamma + \frac{3}{8A}\left\{-\frac{1}{x}e^{-S^2\sin^2\alpha}\int_0^{\infty} c^3 e^{-(c-S\cos\alpha)^2}dc + \frac{1}{2}\left[\frac{\gamma}{x} - \frac{\beta+\gamma}{1-x}\right] + \frac{\beta}{2}\ln\left(\frac{x}{1-x}\right)\right\} + o\left(\frac{1}{A}\right) \tag{6}$$

where β and γ are unknown constants to be determined by the analysis of the end regions of the tube in which the asymptotic solution of Eq. (6) fails to satisfy Eq. (1).

End Regions of the Tube

The analysis of the end region in which x or $1-x = O(1/A)$ is now carried out. In view of the asymptotic solution in most of the tube, the solution up to the order of A^{-1} is assumed in the following form:

$$Q(x) = \beta_* x + \gamma_* + Q^{(0)}(\xi) + \frac{\beta_*}{A}\{Q^{(1)}(\xi) - Q^{(1)}(A-\xi)\} \tag{7}$$

where β_* and γ_* are unknown constants and $Q^{(1)}(\xi)$ is the correction of the impact density arising from the left-hand end, whereas $Q^{(1)}(A-\xi)$ represents the correction in the right-hand end. Further, new constants a and μ related to β_* and γ_* as

$$\beta_* + \gamma_* = -(\beta_*/A)\mu, \quad \gamma_* = a + (\beta_*/A)\mu \tag{8}$$

may be introduced. Substituting Eqs. (7) and (8) in Eq. (1) and equating the same order terms of A, we have (by neglecting higher-order terms in A^{-1})

$$Q^{(0)}(\xi) = N(\xi;S,\alpha) + af_1(\xi) + \int_0^\infty Q^{(0)}(\xi_1) f(|\xi-\xi_1|)\mathrm{d}\xi_1 \tag{9}$$

and we find that the equation for $Q^{(1)}(\xi)$ and μ is the same as Eq. (21) in Ref. 11 if α there is taken to be zero.

From the asymptotic form of $Q(\xi)$ as $\xi\to\infty$ [see Eq. (6)], we have

$$\left.\begin{aligned} &Q^{(0)}(\xi) \to 0 \\ &Q^{(1)}(\xi) \to \frac{3}{16}\ln \xi \end{aligned}\right\} \text{ as } \xi\to\infty \tag{10}$$

The solution given in Eq. (7) should match with the asymptotic solution of Eq. (6) in the region where $\xi \gg 1$ or $A-\xi \gg 1$. This leads to the following relations between constants:

$$\beta+\gamma = -\frac{\beta_*}{A}\left(\mu + \frac{3}{16}\ln A\right), \quad \gamma = a + \frac{\beta_*}{A}\left(\mu + \frac{3}{16}\ln A\right) \tag{11}$$

Accordingly, from Eqs. (8) and (11), we have

$$\beta = a\left\{-1 + \frac{2}{A}\left(\mu + \frac{3}{16}\ln A\right)\right\}; \quad \gamma = a\left\{1 - \frac{1}{A}\left(\mu + \frac{3}{16}\ln A\right)\right\} \tag{11a}$$

Equation (3) of $R(x)$ may be analyzed in the same way. It will not be necessary to repeat the detail of the analysis, but it may be enough to say that if S is set equal to zero in the preceding equations (6)~(11a), these hold for $R(x)$, too. In this case, it may be easily shown[11] that $a=1$.

Pressure Ratio

The number flux F, which may be computed from formula (2a), is now calculated. The main contribution to the integration in this formula arises in the region where $x \sim x_1 \sim ½$ because of the rapidly varying function f in x. Therefore, the number flux can be evaluated with the asymptotic solution in Eq. (6) accompanied with constants of Eqs. (11a). The result, up to $O(A^{-2})$, becomes

$$F = -\frac{4}{3A}\beta = \frac{4}{3A}\left\{1 - \frac{2}{A}\left(\mu + \frac{3}{16}\ln A\right)\right\}a \tag{12}$$

The number flux F_c of the counterflow is also calculated in the same way. The result is easily obtained as $F_c = F[a=1]$. Therefore, Eq. (5) of the pressure ratio becomes

$$p_c/p_0 = (T_c/T_0)^{1/2} a \tag{13}$$

This has the same form as is given by Hughes.[6] The value a in Ref. 6, which is denoted by R in his notation, is a function of S, α, and A in general, whereas the present a does not include A [see Eq. (9)]. This indicates that the pressure ratio is insensitive to A for large A and finite S.

IV. Numerical Calculations

Equations (12) and (13) include constants a and μ. The constant μ has already been given in Ref. 11. The other constant a, in principle, could be obtained by solving the integral Eq. (9) along with the boundary condition Eq. (10). However, as is shown below, this constant can be related to an integral of $Q^{(0)}(\xi)$, to which a variational principle may be applied instead. Introducing a new function

$$g(\xi) = Q^{(0)}(\xi) + a \tag{14}$$

then we get from Eq. (9)

$$g(\xi) = N(\xi;S,\alpha) + \int_0^\infty g(\xi_1) f(|\xi-\xi_1|)\mathrm{d}\xi_1 \tag{15}$$

On multiplying both sides of Eq. (9) by ξ and integrating from 0 to ∞ with respect to ξ, we obtain

$$\frac{a}{3} = \int_0^\infty \xi N(\xi;S,\alpha)\mathrm{d}\xi + \int_0^\infty g(\xi) f_2(\xi)\mathrm{d}\xi \tag{16}$$

These two equations (15) and (16) are suitable for application of a variational method[13] in order to calculate a. The functional $F_0(\theta,\chi)$ in Ref. 13 is used to compute the second integral in Eq. (16). Here, θ and χ are trial functions. Taking into account the asymptotic form of Q as $\xi\to\infty$ [see Eq. (10)], we may assume the following trial functions:

$$\theta = \theta_j = \frac{3}{32}\ln(1+\xi) + \sum_{i=0}^{j} \frac{C_i}{(1+\xi)^i} \tag{17a}$$

$$\chi = \chi_j = -\frac{3}{32}\ln(1+\xi) + \sum_{i=0}^{j} \frac{D_i}{(1+\xi)^i} \tag{17b}$$

where C_i and D_i are variational parameters, which will be determined by the stationary condition. Details of the method may be found in Ref. 13.

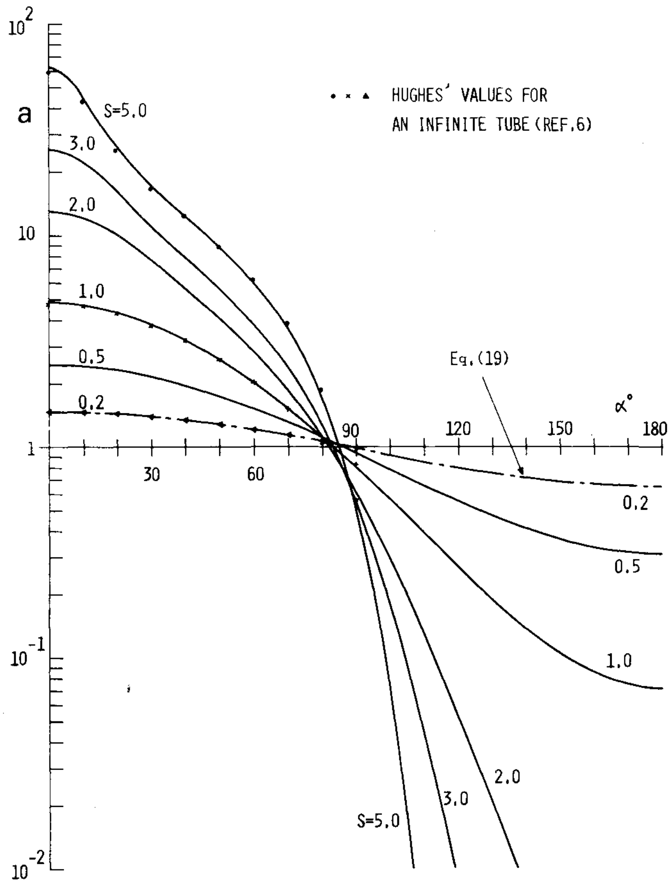

Fig. 2 Values of a vs angle of attack.

Figure 2 shows values of a calculated against the angle of attack for values of S from 0.5 to 5.0. Here, the trial function with $j=3$ were used. Numerical calculations were carried on the FACOM M190 at Kyoto University. Two other cases computed with different pairs of trial fucntions θ_j and χ_j, where $j=1$ and 2 are taken, gave values of a very close to those in Fig. 2. This seems to indicate that the value obtained for $j=3$ is very near the exact value. The results from this study may be compared with Hughes' calcullation[6] in which the Clausing approximation is used. Several values of a for an infinite tube from Ref. 6 are also plotted in Fig. 2. The two theories show a fairly good agreement for small values of S. The difference between them, however, becomes large with increasing S, reaching about 10% at some α's when $S=5.0$. It is expected that the Clausing approximation produces larger error at higher S. Nevertheless, we can say that the analysis based on Clausing's solution gives enough good results for most practical purposes with less numerical work.

V. Case of Small Speed Ratios

Finally, the case of small speed ratio is considered. The nonhomogeneous term in Eq. (9) can be expanded in a power series in S, that is,

$$e^{S^2}N(\xi;S,\alpha) = -f_1(\xi) + \frac{1}{\sqrt{\pi}}N_1(\xi)S\cos\alpha + \{N_2(\xi) - M_2(\xi)\cos 2\alpha\}S^2 + \frac{1}{4\sqrt{\pi}}\{N_3(\xi)\cos\alpha - \frac{5}{3}M_3(\xi)\cos 3\alpha\}S^3 + \ldots \quad (18)$$

where

$$N_1(\xi) = (2Z - Z^{-1})E(Z^{-1}) - 2(Z - Z^{-1})K(Z^{-1}) \quad (18a)$$

$$N_2(\xi) = -\xi + Z - 1/(2Z) + 1/(8Z^3) \quad (18b)$$

$$M_2(\xi) = -\xi + Z - 1/(2Z) - 3/(8Z^3) \quad (18c)$$

$$N_3(\xi) = \{6Z - 7/(2Z) + 1/Z^3\}E(Z^{-1}) + \{-6Z + 13/(2Z) - 1/(2Z^3)\}K(Z^{-1}) \quad (18d)$$

$$M_3(\xi) = \{2Z - 1/(2Z) - 1/Z^3\}E(Z^{-1}) + \{-2Z + 3/(2Z) + 1/(2Z^3)\}K(Z^{-1}) \quad (18e)$$

and $Z=\sqrt{1+\xi^2}$; K and E are the elliptic integrals of the first and second kind, respectively. Equation (18) suggests that a and $Q^{(0)}(\xi)$ may be expanded in series of $S^m\cos n\alpha$ ($^m_n = 0, 1, \cdots$). Substituting these expansions in Eq. (9) and equating the same terms of $S^m\cos n\alpha$, we get integral equations which are essentially the same as Eq. (9). The variational method employed in Sec. IV is applied to evaluate a_{mn} involved in these equations, where

$$e^{S^2}a = \sum_{m,n=0} S^m(\cos n\alpha)a_{mn}$$

Trial functions θ_3 and χ_3 of Eq. (17) were used, and finally the following expression for a is obtained

$$a = \{1 + 2.04025(\cos\alpha)S + (1.63767 + 0.91300\cos 2\alpha)S^2 + (2.19433\cos\alpha + 0.25679\cos 3\alpha)S^3 + \ldots\}e^{-S^2} \quad (19)$$

Other trial functions θ_j and χ_j, when $j=1$ and 2 were taken, gave almost the same a_{mn} as in Eq. (19). The pressure ratio is given by Eqs. (13) and (19). Figure 2 shows a curve calculated from Eq. (19) for $S=0.2$ together with the numerical results obtained by Hughes for an infinite tube.[6] It will be seen that both results are in a good agreement.

VI. Conclusions

Numerical calculations as well as the analytical expression of the pressure in the gage volume for small S reveal that the analysis based on the Clausing approximation gives good results, especially for small S. Its error, however, becomes large with increasing S, reaching about 10% at some angles of attack when $S=5.0$.

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